

The use of GAP and MAPLE software in teaching group theory

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ABSTRACT

Algebra is a subject that must be taken by all students in an undergraduate mathematics study program. However, teachers are often faced with students' difficulties in understanding some of the concepts that are contained in this subject. The tendency of students who generally follow the development of technology can be seen as an opportunity to overcome these problems. Several studies have been undertaken regarding the use of computers in teaching abstract algebra. This paper will detail a comparison between GAP and Maple software as a commonly used software and constantly updated until today. The paper focused on the concept of group theory as the basis for understanding the course of abstract algebra. Result shows that in terms of visualization, Maple is better than GAP but in terms of substance GAP is better than Maple.

Keywords: Abstract, Algebra, GAP, Maple, Group, Theory.

INTRODUCTION

Algebraic structure is one of the compulsory subjects in the Mathematics Study Program curriculum across the University in Indonesia. Therefore, every student in the mathematics study program should take the course. However, a common problem faced by the lecturers of this course is the students' difficulty grasping the concepts that contained in the course. That fact strengthened with students who tend to have low grades for this subject. Hazzan, in his paper, mentioned that many lecturers of the Algebraic Structure course reported student' difficulty in understanding the material they convey. Similar problems are also raised by other researchers such as Dubinsky and Leron (Dubinsky & Leron, 1994). There are some things that caused it, such as: (1) the concepts in Algebraic Structure are very abstract, (2) many examples of the concept are not recognized properly by the students, (3) a lot of students are not familiar with deductive proof (Okur, et al. 2011).

To overcome this problem, there are some efforts that have been made by the lecturer. These include using some modified teaching method to deliver the material and conducting tutorials to make students have better understanding of the concepts. There are some researchers who also conducting a research related to it. Some researchers provide some kinds of tasks related to a particular concept, then analyse the response of the students in answering the task. Dubinsky, Dautermann, Leron and Zazkis focus on the improvement of students' understanding of specific concepts (Zazkis, et al., 1996). In addition to the development of teaching methods, there are some researchers who use the current conditions as a chance to make an effort to improve the students understanding in algebraic structure course. Currently we are in the age of information or computerization era. Almost all aspects of our life use the computer as a medium or tool in doing things. Computer can also be used as a multifunctional tool. Some researchers take advantage of it in delivering the concepts of algebraic structure. In 1976 Gallian uses a computer program written in FORTRAN to investigate a group. Some other researchers use computational programs designed for algebra, including Geissinger who uses the program Exploring Small Group (ESG), O'Bryan and Sherman use the program Cayley, and

Dubisnky and Leron use ISETL program. In addition, there is also a program that is not designed specifically for computational algebra, such as MATLAB (Asiala et al., 1997).

Other studies conducted by Measure et al. in 2011, they make a comparison of the use of some software in learning algebraic structure. They compare the usage of GAP with ISETL, and ESG with FGB (Finite Group Behavior). ISETL, ESG, and FGB are software that quite old and doesn't have active development such as GAP. Therefore, this paper discussed the comparison of two softwares, which are still being developed. This can be used as a learning tool for group theory. These softwares are GAP and Maple. GAP is software that designed specifically for computational algebra, especially group theory. While Maple is software designed for numeric computation. This paper will discuss how the use of such software in learning Algebraic Structure. The advantages and disadvantages in the usage of that software is also discussed.

Group, Algorithm, and Programming (GAP)

Group, Algorithm, and Programming (GAP) is software that is specifically designed for computing the algebraic structure material, especially group theory. GAP is free, open, and extensible software that can be used for computing group theory. GAP was first developed in 1985 at Lehrstuhl D für Mathematik, RWTH Aachen Germany by Joachim Neubüser, et al. The first version of GAP that was introduced to the public is version 2.4 which was released in 1988. In 2008 GAP receive an award from the ACM / SIGSAM Richard Dimick Jenks Memorial Prize as a superior software engineering for computational algebra. Up until now GAP is still being developed at the University of St. Andrews, St. Andrews, Scotland. The latest version of GAP is 4.8.2 that was released on February 2016 (The GAP Group, 2006).

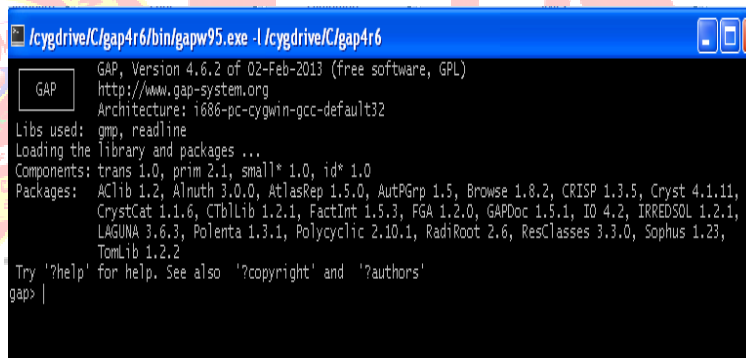


Figure 1: User interface of GAP 4.7.2.

Alexander Hulpke developed GAP installer version 4.4.12. The installer will install GAP and also GGAP, a "graphical user interface" for the system (A. Hulpke, 2011). Figure 2 shows the GGAP user interface.

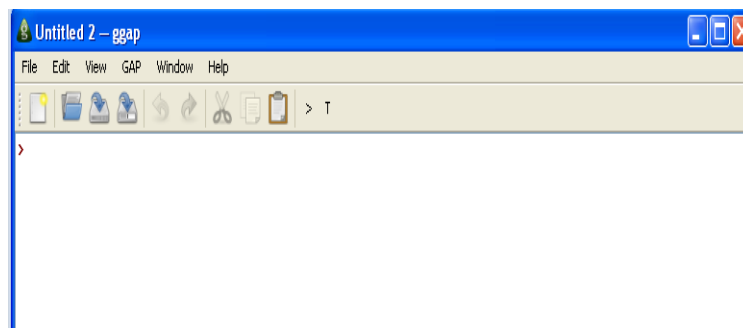


Figure 2: User interface of GGAP.

Although the display of GGAP is more comfortable to be seen, GGAP is still using GAP version 4.4. Therefore, although GGAP can still be used to carry out some specific command, the use of GAP version 4.8.2 is recommended. GAP has approximately 100 packages that acts as an algorithm, a method, or a library. From a programming point of view, this software has a lot of "function" and "operation". GAP currently owns more than 800 default functions that can be used to study various topics in Group Theory. Therefore, GAP can be used to provide many examples, from the simple to complex sample in a relatively short time compared to searching it manually.

Maple

Maple is a commercial computer algebra system developed and sold commercially by Maplesoft, a software company based in Waterloo, Ontario, Canada. The current major version is version 2015, which was released in March 2015. It was first developed in 1980 by the Symbolic Computation Group at the University of Waterloo. The users can enter mathematics in traditional mathematical notation. Custom user interfaces can also be created. There is support for numeric computations, to arbitrary precision, as well as symbolic computation and visualization

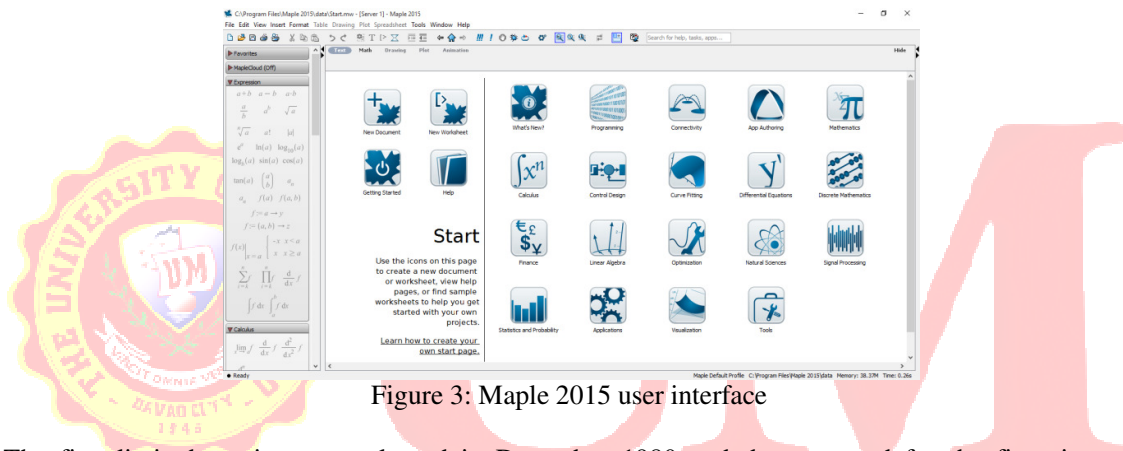


Figure 3: Maple 2015 user interface

The first limited version was released in December 1980 and demonstrated for the first time at conferences in 1982. The name Maple is a reference to Maple's Canadian heritage. Maple is not software that especially designs for computation in algebra, but now it has been equipped with a package that is designed for computational group theory. The package is called Group Theory. The Maple Group Theory package has over 150 commands for constructing groups, examining their properties, performing operations, and visualization. This package includes:

- Large set of group constructors and databases of standard groups so it is easy to get started.
- Support for alternating and symmetric groups; cyclic, dicyclic, and dihedral groups; linear, orthogonal, and unitary groups over finite fields; quaternions; the group of the Rubik's cube; Galois groups; and many more.
- Numerous operations, including isomorphism testing; examining subgroup lattices; splitting a group into conjugacy classes; factoring a group element into a coset representative and a subgroup element; computing if a group is simple, nilpotent, or soluble; identifying isomorphisms for groups up to order 200; computing standard subgroups, such as Sylow subgroups, the center, and the derived subgroup; and many more.
- Easy visualization of Cayley tables and subgroup lattices.
- Symbolic groups, in which some properties of the group are expressed as variable parameters while other characteristics are known.

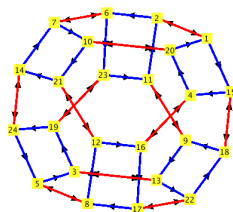


Figure 4. Visualization that can be shown in Maple

The use of Gap in learning group theory

Before using GAP and Maple in the study of Group Theory, the student must first have enough knowledge about the concept of the group theory. First the lecturer gives the definition and example about the concept that will be delivered to the students, and then use GAP and/or Maple to help the students have better understanding about the concept. For example if the concept is quotient group, then the lecturer has to explain first what is Group, Subgroup, and coset in classical way. After the students acquired sufficient knowledge about the group and subgroup, then the lecturer can use GAP and/or Maple to help them understand about the material. This part will describe some steps that can be used to make the students have better understanding in quotient group concept using GAP.

The first step, in GAP worksheet is the students are asked to construct a group. This can be done by trial and error, using the example of a group that has been described in the classroom, or use an existing command on GAP to be able to build a group. Each method has advantages and disadvantages. If students do trial and error in constructing the group, then the students can sharpen their understanding about the group that they have acquired in classroom. GAP will give the output "fail" when the entered set is not a group. This way, students do not only know but also understand the definition of a group. By doing trial and error in constructing the group, then the time required in the process also becomes longer. On the other hand, the process will be faster if the student uses the example of a group that has been delivered by the lecturer or have used the command on GAP for constructing a group. After students have successfully constructed a group, the next step is they will ask to find a subgroup of the group that they have acquired. The next step, the students asked to find all the right and left cosets using GAP, based on the knowledge they have acquired. The students can also see and compare what they have obtained. Figure 5 describes those steps.

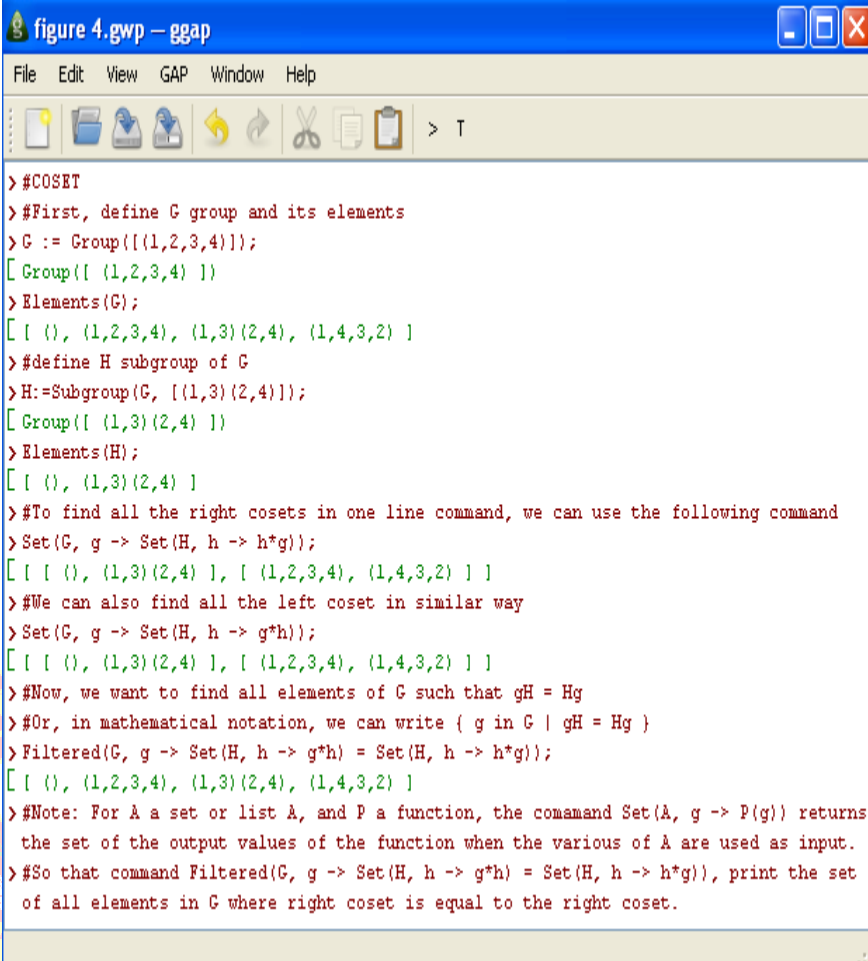
```

>#COSET
>#First, define G group and its elements
>G := Group([(1,2,3,4)]);
[Group([ (1,2,3,4) ])
>Elements(G);
[ [ (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) ]
>#define H subgroup of G
>H:=Subgroup(G, [(1,3)(2,4)]);
[Group([ (1,3)(2,4) ])
>Elements(H);
[ [ (), (1,3)(2,4) ]
>#Now, find left coset (1)H, where (1) is identity
>#Or we can write in mathematical notation, {(1)h | h in H}.
>#Remember that GAP use () to define identity permutation
>Set(H, h -> ()*h);
[ [ (), (1,3)(2,4) ]
>#find another left coset
>Set(H, h -> (1,2,3,4)*h);
[ [ (1,2,3,4), (1,4,3,2) ]
>Set(H, h -> (1,3)(2,4)*h);
[ [ (), (1,3)(2,4) ]
>Set(H, h -> (1,4,3,2)*h);
[ [ (1,2,3,4), (1,4,3,2) ]
>#find all right coset
>Set(H, h -> h*());
[ [ (), (1,3)(2,4) ]
>Set(H, h -> h*(1,2,3,4));
[ [ (1,2,3,4), (1,4,3,2) ]
>Set(H, h -> h*(1,3)(2,4));
[ [ (), (1,3)(2,4) ]
>Set(H, h -> h*(1,4,3,2));
[ [ (1,2,3,4), (1,4,3,2) ]
>

```

Figure 5. Determining left and right coset in GAP worksheet

By using GAP, the student can be trained to find coset of a group in a more pleasant way. That is because they can see firsthand the real shape of cosets they seek. Thus, it is easier for them to understand the concept of coset that was given by the lecturer. It was explained earlier that GAP has many functions that can be used for computational group theory, Figure 6 is one example of that. The picture describes another way that can be used to find all the right and left cosets. It also shows how GAP can check whether the left and right coset is equal or not.



```

>#COSET
>#First, define G group and its elements
>G := Group([(1,2,3,4)]);
[ Group([ (1,2,3,4) ])
>Elements(G);
[ [ (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) ]
>#define H subgroup of G
>H:=Subgroup(G, [(1,3)(2,4)]);
[ Group([ (1,3)(2,4) ])
>Elements(H);
[ [ (), (1,3)(2,4) ]
>#To find all the right cosets in one line command, we can use the following command
>Set(G, g -> Set(H, h -> h*g));
[ [ [ (), (1,3)(2,4) ], [ (1,2,3,4), (1,4,3,2) ] ]
>#We can also find all the left coset in similar way
>Set(G, g -> Set(H, h -> g*h));
[ [ [ (), (1,3)(2,4) ], [ (1,2,3,4), (1,4,3,2) ] ]
>#Now, we want to find all elements of G such that gH = Hg
>#Or, in mathematical notation, we can write { g in G | gH = Hg }
>Filtered(G, g -> Set(H, h -> g*h) = Set(H, h -> h*g));
[ [ (), (1,2,3,4), (1,3)(2,4), (1,4,3,2) ]
>#Note: For A a set or list A, and P a function, the command Set(A, g -> P(g)) returns
the set of the output values of the function when the various of A are used as input.
>#So that command Filtered(G, g -> Set(H, h -> g*h) = Set(H, h -> h*g)), print the set
of all elements in G where right coset is equal to the right coset.

```

Figure 6. Another way to find the left and right coset

Once the students understand how to find left and right coset as well as checking that a coset right and left, the lecturer can provide an explanation of the concept of normal subgroup. Normal subgroup is a group that has the same right and left coset (Gallian, 2010). From the results of their previous work, the definition of normal subgroup is not something new to them. Thus, the students can quickly understand the concept of normal subgroups because the definition is indirectly applied in a practice they already did on the GAP worksheet.

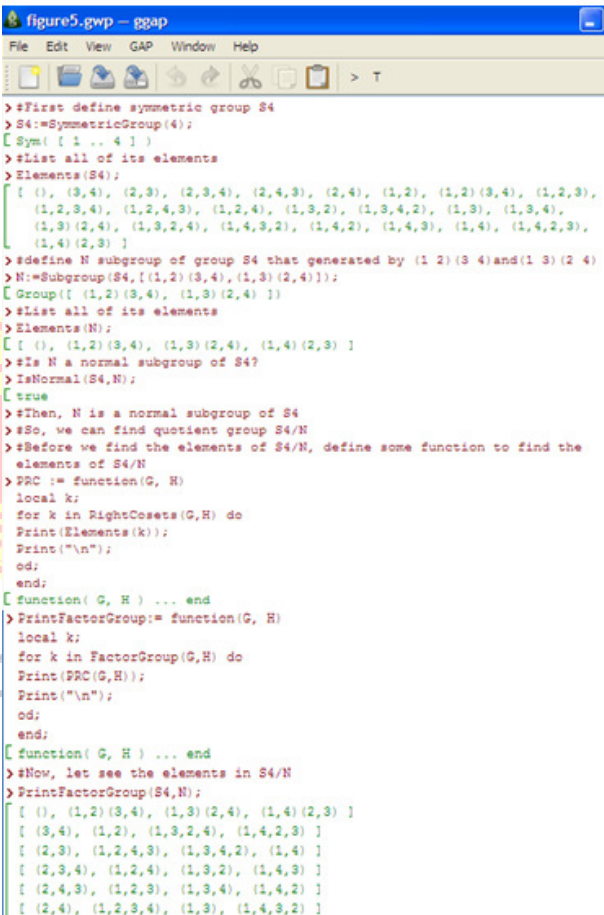
The exercises above can help students to understand what is meant by left and right coset as well as normal subgroup. GAP also helps students to be able to have a concrete idea about the "form" of left and right coset as well as normal subgroup. Thus, they have had enough equipped to face the quotient group material.

To introduce the concept of quotient group G/H , first the lecturer has to show the necessary and sufficient condition for subgroups H so that the operation on G/H is well defined. The necessary and sufficient condition is H should be a normal subgroup (right coset is equal to the left coset) from Group G . After introducing the concept and also its example, sometimes they still can't fully understand how to find the quotient group. Therefore, after working on these examples manually, they can be assigned to do the examples on the GAP worksheet. Figure 5 shows one example of how to obtain quotient groups and elements using GAP. The steps can be described as follow.

For example, define a symmetry group S_4 , then, using commands on GAP, look for all the elements. Furthermore, in a similar way, define N which is a subgroup of the group S_4 generated by $(1\ 2)\ (3\ 4)$

and $(1\ 3)(2\ 4)$. Find all elements of N by using the command in GAP. After that, check if N is a normal subgroup of group S_4 . If N is a normal subgroup of the group S_4 , then write a command on the GAP worksheet to print quotient group S_4 / N . Conversely, if N is not a normal subgroup of the group S_4 , look for other subgroups of the group S_4 which is a normal subgroup.

Thus, by using GAP, the lecturer can introduce the concept to the quotient group among the students in a fun way. By using GAP, students are also trained to increase their creativity, especially in the field of Mathematics. That is because they are trained to apply the concepts of algebra which they have acquired in the form of programming algorithms. They also can easily understand a concept in group theory of that newly acquired, since they can directly see the real form of the concept. Using GAP in learn group theory, can provide a way to learn group theory that is not boring to the students, and consequently can improve students' understanding the concept.



```

>#First define symmetric group S4
>S4:=SymmetricGroup(4);
[Sym( [ 1 .. 4 ] )]
>#List all of its elements
>Elements(S4);
[ ( ), (3,4), (2,3), (2,3,4), (2,4,3), (2,4), (1,2), (1,2)(3,4), (1,2,3),
  (1,2,3,4), (1,2,4,3), (1,2,4), (1,3,2), (1,3,4,2), (1,3), (1,3,4),
  (1,3)(2,4), (1,3,2,4), (1,4,3,2), (1,4,2), (1,4,3), (1,4), (1,4,2,3),
  (1,4)(2,3) ]
>#define N subgroup of group S4 that generated by (1,2)(3,4) and (1,3)(2,4)
>N:=Subgroup(S4,[(1,2)(3,4), (1,3)(2,4)]);
[Group([ (1,2)(3,4), (1,3)(2,4) )]]
>#List all of its elements
>Elements(N);
[ ( ), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) ]
>#Is N a normal subgroup of S4?
>IsNormal(S4,N);
[true]
>#Then, N is a normal subgroup of S4
>#So, we can find quotient group S4/N
>#Before we find the elements of S4/N, define some function to find the
  elements of S4/N
>PRC := function( G, H )
  local k;
  for k in RightCosets(G,H) do
    Print(Elements(k));
    Print("\n");
  od;
end;
>function( G, H ) ... end
>PrintFactorGroup:= function( G, H )
  local k;
  for k in FactorGroup(G,H) do
    Print(PRC(G,H));
    Print("\n");
  od;
end;
>function( G, H ) ... end
>#Now, let see the elements in S4/N
>PrintFactorGroup(S4,N);
[ ( ), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) ]
[ (3,4), (1,2), (1,3,2,4), (1,4,2,3) ]
[ (2,3), (1,2,4,3), (1,3,4,2), (1,4) ]
[ (2,3,4), (1,2,4), (1,3,2), (1,4,3) ]
[ (2,4,3), (1,2,3), (1,3,4), (1,4,2) ]
[ (2,4), (1,2,3,4), (1,3), (1,4,3,2) ]

```

Figure 7. Obtaining factor group and finding its elements using GAP

The use of Maple in learning group theory

Maple has one specific package designed for computation in group theory. The packages are with Group Theory, which has many functions inside it. All the functions that are available are shown in Figure 8.

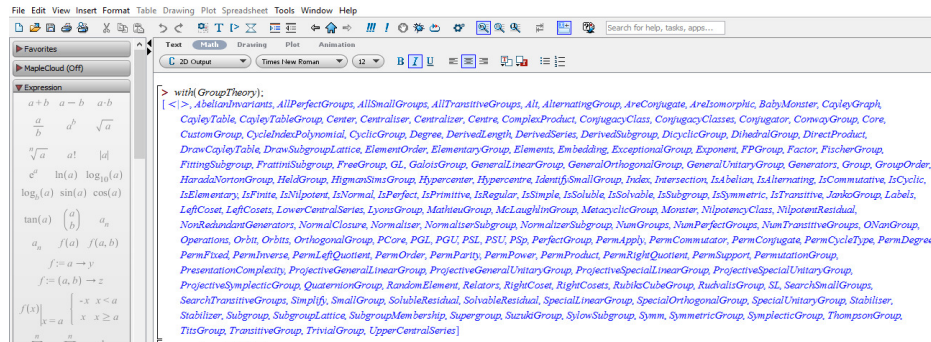


Figure 8. Group theory package

Figure 8 explains that in the package "Group Theory" we can use the following functions

- find the center of a group, the function "center" can be used
- create a dihedral group with a certain degree, the function "DihedralGrup" can be used
- determine the order of a group, the function "GroupOrder" can be used
- determine whether a group is abelian, the function "IsAbelian" can be used
- determine whether a group is cyclic, the function "IsCyclic" can be used
- determine whether a group is a subgroups, the function "IsSubgroup" can be used
- find coset left and right coset, the function "LeftCoset" and "RightCoset", can be used

However, Maple has limitations in computational algebra compared to GAP that is very complete in terms of its package. But in terms of visualization, Maple is more superior compared to GAP. Visualization on GAP is still limited, and can only be used on a particular OS. In terms of appearance, Maple has far more interesting than GAP. Therefore, in this section the discussion is focused on the visualization that can be used on Maple for learning. In the discussion group, students are taught to use the Cayley table to be able them to determine whether a set is a group or not. Suppose a set $S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, table cayley for this set is

	(1)	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1)	(1)	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
(1 2)	(1 2)	(1)	(1 2 3)	(1 3 2)	(1 3)	(2 3)
(1 3)	(1 3)	(1 3 2)	(1)	(1 2 3)	(2 3)	(1 2)
(2 3)	(2 3)	(1 2 3)	(1 3 2)	(1)	(1 2)	(1 3)
(1 2 3)	(1 2 3)	(2 3)	(1 2)	(1 3)	(1 3 2)	(1)
(1 3 2)	(1 3 2)	(1 3)	(2 3)	(1 2)	(1)	(1 2 3)

With the help of maple, Cayley table discussion becomes more enjoyable as maple can produce colored Cayley table. It can be seen in the image below.

	(1)	(1, 2)	(1, 2, 3)	(1, 3)	(2, 3)	(1, 3, 2)
(1)	(1)	(1, 2)	(1, 2, 3)	(1, 3)	(2, 3)	(1, 3, 2)
(1, 2)	(1, 2)	(1)	(1, 3)	(1, 2, 3)	(1, 3, 2)	(2, 3)
(1, 2, 3)	(1, 2, 3)	(2, 3)	(1, 3, 2)	(1, 2)	(1, 3)	(1)
(1, 3)	(1, 3)	(1, 3, 2)	(2, 3)	(1)	(1, 2, 3)	(1, 2)
(2, 3)	(2, 3)	(1, 2, 3)	(1, 2)	(1, 3, 2)	(1)	(1, 3)
(1, 3, 2)	(1, 3, 2)	(1, 3)	(1)	(2, 3)	(1, 2)	(1, 2, 3)

Figure 9. Cayley table of S_3 .

The picture above shows that maple gives a different colour to each of the different elements in the table. It will assist students in identifying the structure of S_3 . Once students understand the Cayley table for S_3 , teachers can improve the set to a set that has more elements, e.g. S_4 . Cayley table for S_4 made in Maple can be seen below.

	e	a	b	c	d	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
e	a	b	c	d	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	
a	a	c	b	f	d	h	g	j	i	k	n	m	p	o	r	q	t	s	v	u	x	w		
b	b	d	g	i	k	n	m	o	q	a	j	s	e	u	w	c	p	l	x	f	t	h	v	r
c	c	f	h	j	l	m	n	p	r	e	i	t	a	v	x	b	o	k	w	d	s	g	u	q
d	d	b	i	g	n	k	o	m	a	q	s	j	u	e	c	w	l	p	f	x	h	t	r	v
f	f	c	j	h	m	l	n	p	r	e	i	t	a	v	x	b	o	k	w	d	s	g	u	q
g	g	k	m	q	j	u	e	w	p	d	a	x	b	t	v	i	c	s	r	n	f	o	h	l
h	h	l	n	r	i	v	a	x	o	f	e	w	c	s	u	j	b	t	q	m	d	p	g	k
i	i	n	o	a	s	e	u	c	l	b	q	f	d	h	r	g	w	j	v	k	x	m	t	p
j	j	m	p	e	t	a	v	b	k	c	r	d	f	g	q	h	x	i	u	l	w	n	s	o
k	k	g	q	m	u	j	w	e	d	p	x	a	t	b	i	v	s	c	n	r	o	f	l	h
l	l	h	r	n	v	i	x	a	f	o	w	e	s	c	j	u	t	b	m	q	p	d	k	g
m	m	j	e	p	a	t	b	v	c	k	d	r	g	f	h	q	i	x	l	u	n	w	o	s
n	n	i	a	o	e	s	c	u	b	l	f	q	h	d	g	r	j	w	k	v	m	x	p	t
o	o	s	u	l	q	h	d	r	w	n	b	v	i	x	t	a	g	f	p	e	k	c	m	j
p	p	t	v	k	r	g	f	q	x	m	c	u	j	w	s	e	h	d	o	a	l	b	n	i
q	q	u	w	d	x	b	t	i	s	g	p	n	k	o	l	m	v	a	h	j	r	e	f	c
r	r	v	x	f	w	e	s	j	t	h	o	m	l	p	k	n	u	e	g	i	q	a	d	b
s	s	o	l	u	h	q	r	d	n	w	v	b	x	i	a	t	f	g	e	p	c	k	j	m
t	t	p	k	v	g	r	q	f	m	x	u	c	w	j	e	s	d	h	a	o	b	l	i	n
u	u	q	d	w	b	x	i	t	g	s	n	p	o	k	m	l	a	v	j	h	e	r	c	f
v	v	r	f	x	e	w	j	s	h	t	m	o	p	l	n	k	e	u	i	g	a	q	b	d
w	w	x	t	s	p	o	k	l	v	u	g	h	q	r	f	d	m	n	c	b	j	i	e	a
x	x	w	s	t	o	p	i	k	u	v	h	g	r	q	d	f	n	m	b	c	i	j	a	e

Figure 10. Cayley table of S_4

In addition to creating the Cayley table, another thing maple can do is to draw a lattice The picture below is one such example.

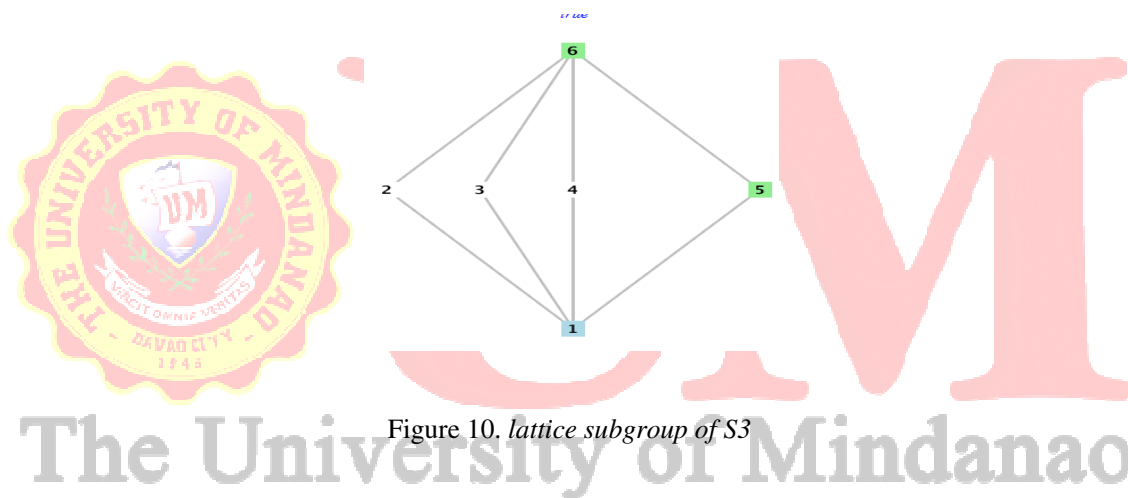


Figure 10. *lattice subgroup of S_3*

From the explanation above, we can see that Maple can generate a nice figure for visualization in Group theory study.

CONCLUSION

In this computerized era, the computer has become a tool that used in almost all aspects of life. Computer has been widely used in educational activities. Thus, in the delivery of the concept of group theory, which on average is difficult to accept by students, the computer can also be used as an educational tool. This paper has described how GAP and Maple can be used as a tool to deliver the concepts of group theory in the classroom. GAP has advantages in terms of complete functions for computational group theory, algebraic structure even more. But the weakness of GAP is in terms of compatibility and its visualization which is still very simple. On the other hand, Maple has advantages in terms of visualization that looks good; it can even be made in the animation. However, the drawback is that Maple limited to a portion of the group theory, but to be used as a tool to help delivering the concept of group theory is still sufficient. Thus, the use of selection in the software adjusted for the needs of each other and can also accompany one another

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